

EXISTENCE OF ROTATING PLANET SOLUTIONS TO THE EULER-POISSON EQUATIONS WITH AN INNER HARD CORE

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ABSTRACT. The Euler-Poisson equations model rotating gaseous stars. Numerous efforts have been made to establish existence and properties of the rotating star solutions. Recent interests in extrasolar planet structures require extension of the model to include a inner rocky core together with its own gravitational potential. In this paper, we discuss various extensions of the classical rotating star results to incorporate a solid core.

1. INTRODUCTION

The motion of a rotating Newtonian gaseous star is described by the following compressible Euler-Poisson equations:

$$(1.1) \quad \begin{cases} \rho_t + \nabla \cdot (\rho \mathbf{v}) = 0 \\ (\rho \mathbf{v})_t + \nabla \cdot (\rho \mathbf{v} \otimes \mathbf{v}) + \nabla p = -\rho \nabla \phi \\ \Delta \phi = 4\pi \rho \end{cases}$$

Here ρ , p , ϕ and \mathbf{v} are respectively density, pressure, gravitational potential and velocity vector field of the gas comprising a star. The solution to the Poisson equation is not unique. One picks the decaying solution at infinity

$$(1.2) \quad -\phi = B\rho = \rho * \frac{1}{|\mathbf{x}|} = \int \frac{\rho(\mathbf{y})}{|\mathbf{x} - \mathbf{y}|} d\mathbf{y}$$

according to Newton's law of gravitation, where $*$ defines a convolution. In order to model a rotating star in dynamical equilibrium, one further makes the assumptions that the solution is axisymmetric and time independent. Under these assumptions the first equation in (1.1) is indentially satisfied, whereas the second equation is reduced to

$$(1.3) \quad \frac{\nabla p}{\rho} = -\nabla \phi + r\Omega^2 \mathbf{e}_r.$$

Here we are assuming that the velocity vector field \mathbf{v} is given by $\mathbf{v} = r\Omega(r)\mathbf{e}_\theta$, where the cylindrical coordinates (r, θ, z) and orthogonal frame field $\{\mathbf{e}_r, \mathbf{e}_\theta, \mathbf{e}_z\}$ are used. To close the system one imposes an equation of state $p = p(\rho)$, sets up the velocity field \mathbf{v} , and prescribes the total mass $\int_{\mathbb{R}^3} \rho d\mathbf{x}$. We seek a non-negative axisymmetric solution ρ to (1.3). The existence and properties of rotating star solutions to (1.3) were attained by Auchmuty and Beals [3], Auchmuty [2], Caffarelli and Friedman [5], Friedman and Turkington [8, 10, 9], Li [11], Chanillo and Li [6], Luo and Smoller [12], and Luo and Smoller [13].

Recent observations on extrasolar giant gaseous planets have raised fundamental questions about their interior structure and origin. Many of the extrasolar gaseous

planets possess unexpectedly small radii, suggesting high metallicity in their composition and possibly the existence of a solid core in the center (Anderson and Adams [1]). Efforts have been made to simulate the evolution of these planets, and evidence for the existence of a solid core has been found (Militzer et al. [14]). Models involving high metallicity and center core have been constructed and examined (Miller et al. [15], Burrows et al. [4]). As a first model from a mathematical perspective, one could modify the Euler-Poisson equations for rotating stars to include a solid core and its gravitational potential. Let K be an axisymmetric bounded domain in \mathbb{R}^3 , and ρ_K be a given axisymmetric non-negative function on K , indicating the density of the solid core. Let $\phi_K = -\rho_K * \frac{1}{|\mathbf{x}|}$ denote the gravitational potential of ρ_K . Then by the $-\phi_K$ -modified Euler-Poisson system we mean the following

$$(1.4) \quad \begin{cases} \rho_t + \nabla \cdot (\rho \mathbf{v}) = 0 \\ (\rho \mathbf{v})_t + \nabla \cdot (\rho \mathbf{v} \otimes \mathbf{v}) + \nabla p = -\rho \nabla(\phi + \phi_K) \\ \Delta \phi = 4\pi \rho \end{cases}$$

As is in the case of rotating star solutions, we assume axisymmetry and time independence to reduce the equations as follows.

$$(1.5) \quad \frac{\nabla p}{\rho} = -\nabla(\phi + \phi_K) + r\Omega^2 \mathbf{e}_r.$$

The goal of the present paper is to discuss existence and non-existence of solutions to (1.5).

The idea that led to existence results for (1.3) is to regard it as a gradient:

$$(1.6) \quad \nabla(a(\rho)) = -\nabla\phi + \nabla J,$$

where

$$(1.7) \quad a(s) = \int_0^s \frac{p'(t)}{t} dt, \text{ and } J(r) = \int_0^r s\Omega^2(s) ds.$$

From (1.6), we get

$$(1.8) \quad a(\rho) = -\phi + J(r) + \lambda$$

for some constant λ . With prescribed equation of state and angular velocity profile $\Omega(r)$, (1.8) is a single equation for the unknown function ρ , although we still don't know the value of λ at this stage.

Now the key idea to solve relation (1.8) is to regard it as the Euler-Lagrange equation of the following energy functional

$$(1.9) \quad E(\rho) = \int_{\mathbb{R}^3} \left(A(\rho) - \frac{1}{2} \rho B \rho - \rho J \right) d\mathbf{x}$$

subject to the constraint

$$(1.10) \quad \int_{\mathbb{R}^3} \rho d\mathbf{x} = M.$$

Here

$$(1.11) \quad A(s) = \int_0^s a(t) dt$$

and

$$(1.12) \quad B\rho(\mathbf{x}) = \rho * \frac{1}{|\mathbf{x}|} = \int \frac{\rho(\mathbf{y})}{|\mathbf{x} - \mathbf{y}|} d\mathbf{y}.$$

Under this formulation the unknown constant λ in (1.8) is naturally realized as a Lagrange multiplier.

Auchmuty and Beals [3] imposed some non-trivial decay conditions on J and got the first existence results for rotating stars. However, these conditions excluded constant Ω and therefore ruled out a large family of physically interesting solutions. Li [11] removed this restriction and was able to obtain an existence result for J with small L^∞ norm. This enabled him to prove existence for small constant Ω . The smallness of $\|J\|_\infty$ is essential in Li's proof. He also substantiated the smallness requirement by proving a non-existence result for (1.8) with large constant Ω .

Following a similar route, one can write down a scalar equation for the modified Euler-Poisson equations:

$$(1.13) \quad a(\rho) = -\phi - \phi_K + J(r) + \lambda.$$

The corresponding Euler-Lagrange energy is

$$(1.14) \quad E(\rho) = \int_{\mathbb{R}^3 \setminus K} \left(A(\rho) - \frac{1}{2} \rho B\rho - \rho J - B\rho_K \right) d\mathbf{x}.$$

Recall that ρ_K is the density of the axisymmetric solid core and is positive. The minus sign in front of the $B\rho_K$ term is slightly surprising. One would expect that the gravitation of an extra center core should somehow cancel out the effect of centrifugal force due to the appearance of J , but at least on the energy level, they are of the same sign. In particular, without a smallness assumption on ρ_K , Li's proof no longer works. It is physically reasonable to assume slow rotation in order for a solution to exist, but unreasonable to assume smallness of solid core density.

Furthermore, with a given rotation profile that is not necessarily small, a solution may still exist if the core gravitational pull is sufficiently large. The potential existence of a fast rotating planet with heavy core is a unique phenomenon that is not present in the classical rotating star case.

On the other hand, with a given core density and total mass, there should still be no solution if the rotation is sufficiently large. One may want to look for a non-existence result like the one in [11]. However, the proof in [11] involves a subtle argument based on integral identities, and fails to apply to the case with a solid core. We thus employ a different argument to show non-existence.

2. STATEMENT OF RESULTS

Let us consider the following axisymmetric equilibrium Φ_K -modified Euler-Poisson equations in $\mathbb{R}^3 \setminus K$, for a bounded axisymmetric domain K :

$$(2.1) \quad \frac{\nabla p}{\rho} = \nabla(B\rho + J + \Phi_K),$$

which is the gradient of the following equilibrium relation

$$(2.2) \quad A'(\rho) - B\rho - J - \Phi_K = \lambda.$$

Here

$$(2.3) \quad B\rho(\mathbf{x}) = \int_{\mathbb{R}^3 \setminus K} \frac{\rho(\mathbf{y})}{|\mathbf{x} - \mathbf{y}|} d\mathbf{y}$$

is the Newtonian potential of ρ ,

$$(2.4) \quad J(r) = \int_0^r s \Omega^2(s) ds,$$

where $r = \sqrt{x_1^2 + x_2^2}$, is the potential of centrifugal force, and Φ_K is the potential generated by the core. We assume

$$(2.5) \quad s \Omega^2(s) \text{ is a given non-negative function in } L^1[0, \infty) \cap C[0, \infty).$$

If gravity is the only effect of the core, Φ_K is given by

$$(2.6) \quad \Phi_K(\mathbf{x}) = B \rho_K(\mathbf{x}) = \int_K \frac{\rho_K(\mathbf{y})}{|\mathbf{x} - \mathbf{y}|} d\mathbf{y}$$

where $\rho_K \in L^q(K)$ for some $q > 3$ is a given axisymmetric non-negative function on K . More generally, Φ_K is a function satisfying

$$(2.7) \quad \Phi_K \in C^1(\mathbb{R}^3) \text{ is positive, axisymmetric, and } \lim_{\mathbf{x} \rightarrow \infty} \Phi_K(\mathbf{x}) = 0,$$

and

$$(2.8) \quad \text{there is a } z_0 > 0 \text{ such that if } |x_3| > z_0, \Phi_K \text{ is non-increasing as } |x_3| \text{ increases.}$$

The equation of state is given by $p = f(\rho)$, where f is a function satisfying

$$(2.9) \quad f(s) \text{ is non-negative, continuous, and strictly increasing for } s > 0.$$

$$(2.10) \quad \lim_{s \rightarrow 0} f(s) s^{-\frac{4}{3}} = 0, \quad \lim_{s \rightarrow \infty} f(s) s^{-\frac{4}{3}} = \infty.$$

A typical example of such an f would be $f(s) = s^\gamma$ for some $\gamma > \frac{4}{3}$. The internal energy potential A in (2.2) is related to f by

$$(2.11) \quad A(s) = s \int_0^s \frac{f(t)}{t^2} dt.$$

We then have the following

Theorem 2.1. *Given $M > 0$, Φ_K satisfying (2.7), and f satisfying (2.9) and (2.10), there is an $\epsilon_1 > 0$, such that if $\|J\|_\infty < \epsilon_1$, there exists a compactly supported axisymmetric continuous function $\rho : \mathbb{R}^3 \setminus K \rightarrow [0, \infty)$, such that*

- (1) ρ is differentiable where it is positive, and satisfies the Φ_K -modified Euler-Poisson equation (2.1) there.

$$(2) \quad \int_{\mathbb{R}^3 \setminus K} \rho(\mathbf{x}) d\mathbf{x} = M.$$

Theorem 2.2. *Given $M > 0$, Φ_K satisfying (2.7), and f satisfying (2.9) and (2.10), there is an $\epsilon_2 > 0$, such that if $\Omega(s) \equiv \Omega < \epsilon_2$, there exists a compactly supported axisymmetric continuous function $\rho : \mathbb{R}^3 \setminus K \rightarrow [0, \infty)$, such that*

- (1) ρ is differentiable where it is positive, and satisfies the Φ_K -modified Euler-Poisson equation (2.1) there.

$$(2) \quad \int_{\mathbb{R}^3 \setminus K} \rho(\mathbf{x}) d\mathbf{x} = M.$$

Theorem 2.1 and Theorem 2.2 establish existence of rotating planet solutions with given mass and core potential for sufficiently small angular velocity profile.

Furthermore, we have the following

Theorem 2.3. *Given $M > 0$, J satisfying (2.5), f satisfying (2.9) and (2.10), and Φ_K satisfying (2.7), there is a $\mu_0 > 0$, such that if $\mu > \mu_0$, there exists a compactly supported axisymmetric continuous function $\rho : \mathbb{R}^3 \setminus C \rightarrow [0, \infty)$, such that*

- (1) ρ is differentiable where it is positive, and satisfies the $\mu\Phi_K$ -modified Euler-Poisson equations there.

(2) $\int_{\mathbb{R}^3 \setminus K} \rho(\mathbf{x}) \, d\mathbf{x} = M.$

Theorem 2.4. *Given $M > 0$, $\Omega(r) \equiv \Omega \geq 0$, f satisfying (2.9) and (2.10), and Φ_K satisfying (2.7), there is a $\mu_0 > 0$, such that if $\mu > \mu_0$, there exists a compactly supported axisymmetric continuous function $\rho : \mathbb{R}^3 \setminus C \rightarrow [0, \infty)$, such that*

- (1) ρ is differentiable where it is positive, and satisfies the $\mu\Phi_K$ -modified Euler-Poisson equations there.

(2) $\int_{\mathbb{R}^3 \setminus K} \rho(\mathbf{x}) \, d\mathbf{x} = M.$

Theorem 2.3 and Theorem 2.4 establish existence of rotating planet solutions with given mass and angular velocity profile for sufficiently large core potential.

Finally, in order to describe a non-existence theorem for fast constant rotation, we need some further assumptions on the equation of state f .

$$(2.12) \quad \liminf_{s \rightarrow \infty} f(s)s^{-\gamma} > 0, \text{ for some } \gamma > \frac{4}{3}.$$

$f(s)$ is continuously differentiable for $s > 0$ and

$$(2.13) \quad \liminf_{s \rightarrow 0} f'(s)s^{-\mu} > 0$$

for some $\mu > 0$. A typical example of such an f is again given by $f(s) = s^\gamma$ for some $\gamma > \frac{4}{3}$.

Theorem 2.5. *Suppose f satisfies (2.9), (2.10), (2.12) and (2.13). Let Φ_K be given by (2.6), and let $M > 0$ be given. Also assume that K satisfies the “no trapping” condition:*

If $(x, y, z) \in \mathbb{R}^3 \setminus K$, then the half line $(x, y, z) + t(x, y, 0)$, ($t \geq 0$) also belongs to $\mathbb{R}^3 \setminus K$.

Then there exists an $\Omega_0 > 0$ such that for $\Omega(r) \equiv \Omega > \Omega_0$, there does not exist a bounded continuous function $\rho : \mathbb{R}^3 \setminus C \rightarrow [0, \infty)$, such that

- (1) ρ satisfies (2.2) where positive.

(2) $\int_{\mathbb{R}^3 \setminus K} \rho(\mathbf{x}) \, d\mathbf{x} = M.$

Theorems 2.1 and 2.2 are proved in section 4. Theorems 2.3 and 2.4 are proved in section 5. Theorem 2.5 is proved in section 6.

3. VARIATIONAL FORMULATION

We first need a few convolution inequalities. These lemmas turn out to be quite useful for the rotating star existence theory. Their proofs can be found in [3].

Lemma 3.1. *Suppose $\rho \in L^1(\mathbb{R}^3) \cap L^p(\mathbb{R}^3)$, and $1 < p \leq \frac{3}{2}$. Then $B\rho \in L^r(\mathbb{R}^3)$ for all $3 < r < \frac{3p}{3-2p}$, and*

$$(3.1) \quad \|B\rho\|_r \leq C(\|\rho\|_1^b \|\rho\|_p^{1-b} + \|\rho\|_1^c \|\rho\|_p^{1-c})$$

for some constant C and $0 < b, c < 1$ depending on p and r . If $p > \frac{3}{2}$, then $B\rho$ is bounded and continuous and satisfies (3.1) with $r = \infty$.

Lemma 3.2. *If $\rho \in L^1(\mathbb{R}^3) \cap L^{4/3}(\mathbb{R}^3)$, then*

$$(3.2) \quad \left| \int_{\mathbb{R}^3} \rho B\rho \, d\mathbf{x} \right| \leq C \left(\int_{\mathbb{R}^3} |\rho|^{4/3} \, d\mathbf{x} \right) \left(\int_{\mathbb{R}^3} |\rho| \, d\mathbf{x} \right)^{2/3}.$$

Lemma 3.3. *If $\rho \in L^1(\mathbb{R}^3) \cap L^p(\mathbb{R}^3)$ for some $p > 3$, then $B\rho$ is continuously differentiable.*

As [3] and [11], we will solve this problem via a variational approach. Let us consider the energy functional

$$(3.3) \quad E(\rho) = \int_{\mathbb{R}^3 \setminus K} \left(A(\rho)(\mathbf{x}) - \frac{1}{2} \rho(\mathbf{x}) B\rho(\mathbf{x}) - \rho(\mathbf{x}) J(\mathbf{x}) - \rho(\mathbf{x}) \Phi_K(\mathbf{x}) \right) d\mathbf{x},$$

where A is given by (2.11), on the space of admissible functions

$$W = \left\{ \rho : \mathbb{R}^3 \setminus K \rightarrow \mathbb{R}, \rho \text{ is axisymmetric, } \rho \geq 0 \text{ a.e., } \int_{\mathbb{R}^3 \setminus K} A(\rho) < \infty, \int_{\mathbb{R}^3 \setminus K} \rho = M \right\}.$$

We first verify that E is well-defined on W . From (2.10), it follows easily that

$$(3.4) \quad \lim_{s \rightarrow 0} A(s) s^{-\frac{4}{3}} = 0, \quad \lim_{s \rightarrow \infty} A(s) s^{-\frac{4}{3}} = \infty.$$

(2.9) and (3.4) imply the existence of a $c > 0$ such that

$$(3.5) \quad A(s) \geq cs^{4/3}$$

for $s > 1$. Hence

$$(3.6) \quad \begin{aligned} \int \rho^{4/3} &\leq \frac{1}{c} \int A(\rho) + \int_{\rho < 1} \rho^{4/3} \\ &\leq \frac{1}{c} \int A(\rho) + M. \end{aligned}$$

(3.6) and lemma 3.2 give the finiteness of the second term in (3.3). The last two terms in (3.3) are finite because J and Φ_K are bounded functions. We have shown that E is well-defined on W .

The basic assertion is the following:

Proposition 3.4. *If ρ is a local minimum for E in W , then ρ is continuous and is differentiable where it is positive, and satisfies (2.1) there.*

Proof. The proof is standard. See [3]. □

4. EXISTENCE FOR SLOW ROTATION WITH FIXED CORE DENSITY

In the following proof, we will construct a number of bounds R_n on the size of the support of the density functions. Without further mentioning, we always assume that R_{n+1} is no less than R_n . All constants in the following may depend on M , f , $\|J\|_\infty$ and Φ_K . Cartesian coordinates $\mathbf{x} = (x_1, x_2, x_3)$ and cylindrical coordinates (r, θ, z) are used interchangeably. To look for a minimizer of E in W , let us first show that E is bounded from below.

Proposition 4.1. *There is a $C > 0$ such that $E(\rho) \geq -C$ for all $\rho \in W$.*

Proof. By lemma 3.2, we have

$$(4.1) \quad E(\rho) \geq \int A(\rho) \, d\mathbf{x} - M\|J + \Phi_K\|_\infty - \frac{1}{2}cM^{2/3} \int \rho^{4/3} \, d\mathbf{x}.$$

By (3.4), there is an $s > 0$ such that for $\rho > s$, $A(\rho) > \frac{1}{2}cM^{2/3}\rho^{4/3}$. Therefore

$$\begin{aligned} E(\rho) &\geq \int_{\rho > s} A(\rho) \, d\mathbf{x} - M\|J + \Phi_K\|_\infty - \frac{1}{2}cM^{2/3} \int_{\rho > s} \rho^{4/3} - \frac{1}{2}cM^{2/3}s^{1/3} \int_{\rho < s} \rho \, d\mathbf{x} \\ &\geq -M\|J + \Phi_K\|_\infty - \frac{1}{2}cM^{5/3}s^{1/3}. \end{aligned}$$

□

Now that we know E is bounded from below, it makes sense to talk about the infimum of E . Let

$$(4.2) \quad I = \inf_{\rho \in W} E(\rho).$$

We will take a sequence of minimizers in bounded balls as a minimizing sequence for I . For that purpose, we need to define

$$(4.3) \quad W_R = \left\{ \rho \in W \mid \text{Supp } \rho \in S_R, \, 0 \leq \rho \leq R \text{ a.e.} \right\}.$$

Here S_R is the ball centered at the origin with radius $R > R_0$ so large that K is contained in S_R . As usual we will extend functions in W_R by zero values outside S_R , and treat them as functions defined on the whole space if necessary. The next assertion is the starting point of this existence method.

Proposition 4.2. *There is an $R_0 > 0$ such that for $R > R_0$, there exists some $\rho_R \in W_R$ which minimizes E :*

$$(4.4) \quad I_R = E(\rho_R) = \inf_{\rho \in W_R} E(\rho).$$

Proof. The proof is standard. See [3] or [11].

□

As in [11], we can give a uniform L^∞ bound on ρ_R .

Lemma 4.3. *There is a $C > 0$, such that*

$$(4.5) \quad \|\rho_R\|_\infty \leq C$$

for all $R \geq R_0$.

Proof. Notice that $\Phi_K \in L^\infty(\mathbb{R}^3)$. The proof in this case is basically the same as that in [11].

□

The L^∞ bound frees the restriction on ρ_R from above, and therefore implies a variational inequality in one direction:

Lemma 4.4. *There is an $R_1 > 0$, such that for all $R > R_1$, there exists a λ_R such that*

$$(4.6) \quad A'(\rho_R) - B\rho_R - J - \Phi_K \geq \lambda_R, \quad \text{in } B_R,$$

$$(4.7) \quad A'(\rho_R) - B\rho_R - J - \Phi_K = \lambda_R, \quad \text{where } \rho_R > 0.$$

Proof. See [3]. □

Lemma 4.5. *There is an $R_2 > 0$ and $e_1 > 0$, such that $I_R \leq -e_1$ for all $R > R_2$.*

Proof. Let

$$(4.8) \quad F(\rho) = \int_{\mathbb{R}^3 \setminus K} \left(A(\rho)(\mathbf{x}) - \frac{1}{2} \rho(\mathbf{x}) B \rho(\mathbf{x}) \right) d\mathbf{x}.$$

This is the corresponding energy functional for an Euler-Poisson system with no rotation and a zero density core. The method in [3] is fully applicable to this case. We therefore get a compactly supported minimizer $\sigma \in W$ of F . Let

$$(4.9) \quad e_1 = -F(\sigma) = -\inf_{\rho \in W} F(\rho).$$

e_1 is seen to be positive by the following scaling argument: pick a non zero $\rho \in W$ that is bounded and compactly supported in $\mathbb{R}^3 \setminus S_{\tilde{R}}$ for some $S_{\tilde{R}} \supset K$. Let

$$(4.10) \quad \rho_t(\mathbf{x}) = t^{-3} \rho(t^{-1} \mathbf{x})$$

for $t > 1$. We see easily that ρ_t is supported in $\mathbb{R}^3 \setminus tB_{\tilde{R}}$, and therefore belongs to W .

$$\begin{aligned} F(\rho_t) &= \int_{\mathbb{R}^3 \setminus tB_{\tilde{R}}} A(\rho_t) - \frac{1}{2} \rho_t B \rho_t \\ &= \int_{\mathbb{R}^3 \setminus B_{\tilde{R}}} (t^3 A(t^{-3} \rho) - \frac{1}{2} t^{-1} \rho B \rho) \\ &= \int_{\text{Supp } \rho} o(t^{-4} \|\rho\|_\infty) t^3 - t^{-1} \frac{1}{2} \int \rho B \rho \\ &= o(t^{-1}) - \Theta(t^{-1}). \end{aligned}$$

The penultimate step follows from (3.4). This shows that the minimum of F must be negative. Now let R_2 be large enough to contain the support of σ , then $\sigma \in W_R$ for $R > R_2$, and

$$\begin{aligned} E(\rho_R) &\leq E(\sigma) \\ &= \int (A(\sigma) - \frac{1}{2} \sigma B \sigma - J \sigma - \Phi_K \sigma) \\ &\leq \int (A(\sigma) - \frac{1}{2} \sigma B \sigma) \\ &= F(\sigma) \\ &= -e_1. \end{aligned}$$

□

Lemma 4.6. *Suppose $\|J\|_\infty < \frac{e_1}{2M}$. There is an $\epsilon_0 > 0$ and an $R_2 > 0$ such that for all $R > R_2$, $\epsilon_R := \sup_{\mathbf{x} \in \mathbb{R}^3} \int_{|\mathbf{x}-\mathbf{y}| < 1} \rho_R(\mathbf{y}) d\mathbf{y} \geq \epsilon_0$.*

Proof. Under the assumption on $\|J\|_\infty$

$$\begin{aligned} & \int \frac{1}{2} \rho_R B \rho_R + \rho_R \Phi_K \\ &= -E(\rho_R) + \int A(\rho_R) - \rho_R J \\ &\geq e_1 - \|J\|_\infty M \\ &\geq \frac{e_1}{2}. \end{aligned}$$

Therefore either

$$(4.11) \quad \int \frac{1}{2} \rho_R B \rho_R \geq \frac{e_1}{4},$$

or

$$(4.12) \quad \int \rho_R \Phi_K \geq \frac{e_1}{4}.$$

If (4.11) happens, then

$$(4.13) \quad \frac{e_1}{2} \leq \int \rho_R B \rho_R \leq M \|B \rho_R\|_\infty.$$

Now

$$\begin{aligned} B \rho_R(\mathbf{x}) &= \int_{\mathbb{R}^3} \frac{\rho_R(\mathbf{y})}{|\mathbf{x} - \mathbf{y}|} d\mathbf{y} \\ &= \int_{|\mathbf{y}-\mathbf{x}| < 1} \frac{\rho_R(\mathbf{y})}{|\mathbf{x} - \mathbf{y}|} d\mathbf{y} + \int_{1 < |\mathbf{y}-\mathbf{x}| < \tilde{R}} \frac{\rho_R(\mathbf{y})}{|\mathbf{x} - \mathbf{y}|} d\mathbf{y} + \int_{|\mathbf{y}-\mathbf{x}| > \tilde{R}} \frac{\rho_R(\mathbf{y})}{|\mathbf{x} - \mathbf{y}|} d\mathbf{y} \\ &:= B_1 + B_2 + B_3. \end{aligned}$$

By lemma 4.3 and lemma 3.1, we have

$$(4.14) \quad B_1 \leq C(\epsilon_R^b + \epsilon_R^c)$$

for some $0 < b, c < 1$. The annulus $1 < |\mathbf{y} - \mathbf{x}| < \tilde{R}$ can be covered by $C\tilde{R}^3$ balls of radius one, hence

$$(4.15) \quad B_2 \leq C\tilde{R}^3 \epsilon_R.$$

One clearly has

$$(4.16) \quad B_3 \leq \frac{M}{\tilde{R}}.$$

Hence

$$(4.17) \quad \|B \rho_R\|_\infty \leq C(\epsilon_R^b + \epsilon_R^c) + C\tilde{R}^3 \epsilon_R + \frac{M}{\tilde{R}}.$$

Choosing \tilde{R} sufficiently large and comparing (4.13) with (4.17), we see that there must be an $\epsilon_0 > 0$ such that $\epsilon_R > \epsilon_0$. Now let us assume that (4.12) happens. We

have

$$\begin{aligned} & \int \rho_R \Phi_K \\ &= \int_{|\mathbf{x}| > \tilde{R}} \rho_R(\mathbf{x}) \Phi_K(\mathbf{x}) d\mathbf{x} + \int_{|\mathbf{x}| < \tilde{R}} \rho_R(\mathbf{x}) \Phi_K(\mathbf{x}) d\mathbf{x} \\ &:= B_1 + B_2. \end{aligned}$$

By (2.7), we can choose \tilde{R} so large that $\Phi_K(\mathbf{x}) \leq \frac{e_1}{8M}$ when $|\mathbf{x}| > \tilde{R}$. Then

$$(4.18) \quad B_1 \leq \frac{e_1}{8}.$$

The ball $|\mathbf{x}| < \tilde{R}$ can be covered by $C\tilde{R}^3$ balls of radius one, hence

$$(4.19) \quad B_2 \leq C\tilde{R}^3 \epsilon_R.$$

Therefore

$$(4.20) \quad \int \rho_R \Phi_K \leq \frac{e_1}{8} + C\tilde{R}^3 \epsilon_R.$$

Comparing (4.12) with (4.20), we again see that such an ϵ_0 exists. \square

Lemma 4.7. *There is an $R_a > 0$ such that if*

$$(4.21) \quad \int_{|\mathbf{y}-\mathbf{x}| < 1} \rho_R(\mathbf{y}) d\mathbf{y} \geq \frac{\epsilon_0}{2},$$

then $r(\mathbf{x}) \leq R_a$. Here $r(\mathbf{x}) = \sqrt{x_1^2 + x_2^2}$.

Proof. Assume $|r(\mathbf{x})| > \tilde{R} + 1$ where $S_{\tilde{R}} \supset K$. By the axisymmetry of ρ_R ,

$$\begin{aligned} Cr(\mathbf{x}) \frac{\epsilon_0}{2} &\leq \int_T \rho_R \leq M, \\ r(\mathbf{x}) &\leq \frac{2M}{C\epsilon_0}. \end{aligned}$$

Here T is the torus obtained from rotating the the ball $|\mathbf{y} - \mathbf{x}| < 1$ around the z -axis. \square

Lemma 4.8. *Suppose $\|J\|_\infty \leq \frac{e_1}{2M}$. There is an $R_3 > R_a$ and an $e_2 > 0$ such that $\lambda_R \leq -e_2$ for all $R > R_2$.*

Proof. By lemma 4.6, for $R > R_2$ there is an \mathbf{x}_R such that

$$(4.22) \quad \int_{|\mathbf{y}-\mathbf{x}_R| < 1} \rho_R(\mathbf{y}) d\mathbf{y} \geq \frac{\epsilon_0}{2}.$$

By lemma 4.7, $r(\mathbf{x}_R) < R_a$. Let \mathbf{x}_0 be on the z -axis such that $z(\mathbf{x}_0) = z(\mathbf{x}_R)$. Let $B(\mathbf{x}_0, R_3)$ be the ball centered at \mathbf{x}_0 with radius $R_3 > R_a$ to be determined. When $R > R_3$, the volume of the set $B(\mathbf{x}_0, R_3) \cap B_R$ is of order R_3^3 . There must exist a point $\mathbf{x} \in B(\mathbf{x}_0, R_3) \cap B_R$ such that

$$(4.23) \quad \rho_R(\mathbf{x}) \leq \frac{CM}{R_3^3}$$

for some constant $C > 0$. Clearly

$$(4.24) \quad |\mathbf{x} - \mathbf{x}_R| \leq |\mathbf{x} - \mathbf{x}_0| + |\mathbf{x}_R - \mathbf{x}_0| \leq 2R_3.$$

Hence

$$(4.25) \quad B\rho_R(\mathbf{x}) \geq \int_{|\mathbf{y}-\mathbf{x}_R|<1} \frac{\rho_R(\mathbf{y})}{|\mathbf{x}-\mathbf{y}|} d\mathbf{y} \geq \frac{1}{2R_3+1} \frac{\epsilon_0}{2}.$$

By (4.6),

$$(4.26) \quad \lambda_R \leq A'(\frac{CM}{R_3^3}) - \frac{1}{2R_3+1} \frac{\epsilon_0}{2}$$

Notice that (2.10) implies

$$(4.27) \quad \lim_{s \rightarrow 0} \frac{A'(s)}{s^{1/3}} = 0.$$

Hence (4.26) implies

$$(4.28) \quad \lambda_R \leq o(R_3^{-1}) - \Theta(R_3^{-1}).$$

Pick R_3 so large that the right hand side of (4.28) becomes negative, and call that $-e_2$. □

Lemma 4.9. *Suppose $\|J\|_\infty \leq \min\left\{\frac{e_1}{2M}, \frac{e_2}{2}\right\}$, then*

$$(4.29) \quad B\rho_R + \Phi_K \geq \frac{e_2}{2} \quad \text{where } \rho_R > 0$$

for $R > R_3$.

Proof. By (4.7) and lemma 4.8, we have

$$(4.30) \quad A'(\rho_R) - B\rho_R - J - \Phi_K = \lambda_R \leq -e_2$$

when $\rho_R > 0$. □

Lemma 4.10. *Suppose $\|J\|_\infty \leq \min\left\{\frac{e_1}{2M}, \frac{e_2}{2}\right\}$. There exists an $R_4 > 0$ such that $\rho_R(\mathbf{x}) = 0$ if $R > r(\mathbf{x}) > R_4$.*

Proof. We have

$$\begin{aligned} B\rho_R(\mathbf{x}) &= \int \frac{\rho_R(\mathbf{y})}{|\mathbf{x}-\mathbf{y}|} d\mathbf{y} \\ &= \int_{|\mathbf{x}-\mathbf{y}|<1} \frac{\rho_R(\mathbf{y})}{|\mathbf{x}-\mathbf{y}|} d\mathbf{y} + \int_{1<|\mathbf{x}-\mathbf{y}|<\tilde{R}} \frac{\rho_R(\mathbf{y})}{|\mathbf{x}-\mathbf{y}|} d\mathbf{y} + \int_{|\mathbf{x}-\mathbf{y}|>\tilde{R}} \frac{\rho_R(\mathbf{y})}{|\mathbf{x}-\mathbf{y}|} d\mathbf{y} \\ &:= B_1 + B_2 + B_3. \end{aligned}$$

Clearly

$$(4.31) \quad B_3 \leq \frac{M}{\tilde{R}}.$$

We choose $\tilde{R} > \frac{12M}{e_2}$, so that

$$(4.32) \quad B_3 < \frac{e_2}{12}.$$

By lemma 3.1,

$$(4.33) \quad B_1 \leq c_0 \left(\left(\int_{|\mathbf{x}-\mathbf{y}|<1} \rho_R(\mathbf{y}) d\mathbf{y} \right)^b + \left(\int_{|\mathbf{x}-\mathbf{y}|<1} \rho_R(\mathbf{y}) d\mathbf{y} \right)^c \right)$$

for some $0 < b, c < 1$. By requiring $R > r(\mathbf{x}) > R_4$ to be large enough, we have

$$(4.34) \quad B_1 \leq c_0 \left(\left(\frac{CM}{R_4} \right)^b + \left(\frac{CM}{R_4} \right)^c \right) < \frac{e_2}{12}$$

by axisymmetry, just like in lemma 4.7. The annulus $1 < |\mathbf{x} - \mathbf{y}| < \tilde{R}$ can be covered by $C\tilde{R}^3$ balls of radius 1. Again by axisymmetry, we have

$$(4.35) \quad B_2 \leq \frac{C\tilde{R}^3 M}{R_4 - \tilde{R}} < \frac{e_2}{12},$$

provided R_4 is chosen to be sufficiently large. Therefore

$$(4.36) \quad B\rho_R(\mathbf{x}) = B_1 + B_2 + B_3 < \frac{e_2}{4}$$

if $R > r(\mathbf{x}) > R_4$. Enlarge R_4 if necessary so that $\Phi_K(\mathbf{x}) < \frac{e_2}{4}$ when $r(\mathbf{x}) > R_4$. We get

$$(4.37) \quad B\rho_R(\mathbf{x}) + \Phi_K(\mathbf{x}) < \frac{e_2}{4} + \frac{e_2}{4} = \frac{e_2}{2}$$

when $R > r(\mathbf{x}) > R_4$. Comparing (4.37) with (4.29), we see that the assertion is true. \square

Lemma 4.11. *Suppose $\|J\|_\infty \leq \min \left\{ \frac{e_1}{2M}, \frac{e_2}{2} \right\}$. There exist $R_5 > 0$, $\delta > 0$ and $r > 0$ such that if $R > z(\mathbf{x}) > R_5$, and if*

$$(4.38) \quad \int_{|z(\mathbf{x}) - z_0| < r} \rho_R(\mathbf{x}) d\mathbf{x} < \delta,$$

then $\rho(\mathbf{x}) = 0$ for $|z(\mathbf{x}) - z_0| < 1$.

Proof. Suppose $r > 2$. If $|z(\mathbf{x}) - z_0| < 1$, $\text{dist}(\mathbf{x}, \{\mathbf{y} \mid |z(\mathbf{y}) - z_0| > r\}) > r - 1$. Just like in lemma 4.10, we have

$$\begin{aligned} B\rho_R(\mathbf{x}) &= \int_{|z(\mathbf{y}) - z_0| < r} \frac{\rho_R(\mathbf{y})}{|\mathbf{x} - \mathbf{y}|} d\mathbf{y} + \int_{|z(\mathbf{y}) - z_0| > r} \frac{\rho_R(\mathbf{y})}{|\mathbf{x} - \mathbf{y}|} d\mathbf{y} \\ &\leq C(\delta^b + \delta^c) + \frac{M}{r-1} \\ &< \frac{e_2}{4} \end{aligned}$$

by choosing δ small and r large. Furthermore $\Phi_K(\mathbf{x}) < \frac{e_2}{4}$ if $z(\mathbf{x}) > R_5$ is sufficiently large. These imply

$$(4.39) \quad B\rho_R(\mathbf{x}) + \Phi_K(\mathbf{x}) < \frac{e_2}{2}.$$

The assertion follows again from a comparison with (4.29). \square

Lemma 4.12. *Suppose $\|J\|_\infty \leq \min \left\{ \frac{e_1}{2M}, \frac{e_2}{2} \right\}$. There is an $R_6 > 0$ such that $\rho_R(\mathbf{x}) = 0$ if $R > z(\mathbf{x}) > R_6$.*

Proof. Let $Z_n = \{x : |z(x) - 2n| < 1\}$, $n = \pm([R_5] + 1), \pm([R_5] + 2), \dots$, and let $Z'_n = \{x : |z(x) - 2n| < r\}$. By lemma 4.11, if ρ_R is not identically zero on a Z_n , then $\int_{Z'_n} \rho_R \geq \delta$. Let m be the number of such n 's. Since each point in \mathbb{R}^3 is covered by at most r different Z'_n 's, $m\delta \leq rM$. Also such Z_n 's must be contiguous, if they lie in the region $|z| > z_0 + 2$ for z_0 given in (2.8). Otherwise there would be an “empty” Z_n below a “non-empty” half space. If one slides the whole “non-empty” half space down by two units to create a new ρ'_R , $\int A(\rho_R) - J\rho_R = \int A(\rho'_R) - J\rho'_R$, but $\int -\frac{1}{2}\rho_R B\rho_R - \rho_R \Phi_K > \int -\frac{1}{2}\rho'_R B\rho'_R - \rho'_R \Phi_K$. This implies $E(\rho_R) > E(\rho'_R)$, but $\rho'_R \in W_R$, a contradiction. Now pick $R_6 > 2\left([R_5] + \frac{rM}{\delta}\right) + z_0 + 3$. The proof is complete. \square

We are now in a position to prove theorem 2.1 and theorem 2.2.

Proof of theorem 2.1. Let $\epsilon_1 = \min\left\{\frac{\epsilon_1}{2M}, \frac{\epsilon_2}{2}\right\}$. From lemma 4.10 and lemma 4.12, we see that $\rho_R = \rho_{R_7}$ when $R > R_7 := \sqrt{2}R_6$. Since $\Phi_K \in L^\infty(\mathbb{R}^3)$, a similar argument as in [3] shows that $\rho = \rho_{R_7}$ minimizes E in W . By proposition 3.4, ρ solves (2.1) and has the stated properties. \square

Proof of theorem 2.2. Let $\epsilon_2 = \frac{\sqrt{\epsilon_1}}{R_7}$, and let $\tilde{J}(r) \in C^\infty(0, \infty)$ be an increasing function such that

$$(4.40) \quad \tilde{J}(r) = \begin{cases} \frac{1}{2}\Omega^2 r^2 & \text{if } r \leq R_7, \\ \Omega^2 R_7^2 & \text{if } r \geq 2R_7. \end{cases}$$

If $\Omega < \epsilon_2$, we have $\|\tilde{J}\| < \epsilon_1$, hence by theorem 2.1, there is a solution ρ to (2.1) where J is replaced by \tilde{J} , supported in S_{R_7} . Clearly such a ρ also solves (2.1) with the original J , and has the stated properties. \square

5. EXISTENCE FOR FAST ROTATION WITH HEAVY CORE DENSITY

In this section, we give proofs to theorem 2.3 and theorem 2.4. That corresponds to establishing existence of minimizer of

$$(5.1) \quad E_\mu(\rho) = \int_{\mathbb{R}^3 \setminus C} \left(A(\rho)(\mathbf{x}) - \frac{1}{2}\rho(\mathbf{x})B\rho(\mathbf{x}) - \rho(\mathbf{x})J(\mathbf{x}) - \mu\rho(\mathbf{x})\Phi_K(\mathbf{x}) \right) d\mathbf{x}$$

for large enough μ . We will omit an argument in the proof if it runs parallel to the proof in the previous section.

As before, E_μ is bounded from below on W and has an infimum which we denote by I_μ . If we pick

$$(5.2) \quad W_R = \left\{ \rho \in W \mid \text{Supp } \rho \in S_R, \rho \geq 0 \text{ a.e.} \right\}.$$

E_μ will also attain its infimum $I_{\mu,R}$ on each W_R . We still denote the minimizers by ρ_R . It is understood that ρ_R implicitly depends on μ . Comparing (4.3) with (5.2), we see that the L^∞ bound on W_R (namely, the $\leq R$ constraint) is removed. This

is to allow large ρ_R on B_R . As we will see later, the L^∞ bound of ρ_R depends on μ and J . For that purpose, we start by modifying the bound on $\|\rho\|_{4/3}$.

Lemma 5.1. *Let ρ_R be a minimizer of E_μ in W_R , and assume that B_{R_0} contains the core K . There is a constant C depending only on f , M , J and Φ_K such that*

$$(5.3) \quad \int \rho_R^{\frac{4}{3}} d\mathbf{x} \leq C(1 + \mu)$$

for all $R > R_0$.

Proof. Let ρ_0 be some fixed function in W_{R_0} . For $R > R_0$,

$$\begin{aligned} & \int \left(A(\rho_0) - \rho_0 J - \frac{1}{2} \rho_0 B \rho_0 - \mu \rho_0 \Phi_K \right) d\mathbf{x} \\ & \geq \int \left(A(\rho_R) - \rho_R J - \frac{1}{2} \rho_R B \rho_R - \mu \rho_R \Phi_K \right) d\mathbf{x} \\ & \geq \int \left(A(\rho_R) - \rho_R (J + \mu \Phi_K) \right) d\mathbf{x} - CM^{\frac{2}{3}} \int \rho_R^{\frac{4}{3}} d\mathbf{x}. \end{aligned}$$

The last step follows from lemma 3.2. By condition (2.10), there is an $s_1 > 0$ such that

$$(5.4) \quad A(s)s^{-\frac{4}{3}} > 2CM^{\frac{2}{3}}$$

for $s > s_1$. Therefore

$$\begin{aligned} \tilde{C} &= \int \left(A(\rho_0) - \rho_0 J - \frac{1}{2} \rho_0 B \rho_0 - \mu \rho_0 \Phi_K \right) d\mathbf{x} \\ &\geq \int_{\rho_R \leq s_1} A(\rho_R) d\mathbf{x} + \int_{\rho_R > s_1} A(\rho_R) d\mathbf{x} - M(\|J\|_\infty + \mu\|\Phi_K\|_\infty) \\ &\quad - CM^{\frac{2}{3}} s_1^{\frac{1}{3}} M - \int_{\rho_R > s_1} \frac{1}{2} A(\rho_R) d\mathbf{x} \\ &\geq \frac{1}{2} \int A(\rho_R) d\mathbf{x} - C'(1 + \mu). \end{aligned}$$

Or,

$$(5.5) \quad \int A(\rho_R) d\mathbf{x} \leq C(M, s_1)(1 + \mu).$$

Notice that we have

$$\begin{aligned} \int \rho_R^{\frac{4}{3}} d\mathbf{x} &= \int_{\rho_R \leq s_1} \rho_R^{\frac{4}{3}} d\mathbf{x} + \int_{\rho_R > s_1} \rho_R^{\frac{4}{3}} d\mathbf{x} \\ &\leq s_1^{\frac{1}{3}} M + \frac{1}{2CM^{\frac{2}{3}}} \int_{\rho_R > s_1} A(\rho_R) d\mathbf{x} \\ &\leq C(M, s_1) \left(1 + \int A(\rho_R) d\mathbf{x} \right). \end{aligned}$$

The assertion is now apparent. \square

Now let us give an L^∞ bound on ρ_R . It is crucial to make the power of μ as low as possible.

Lemma 5.2. *There is an $R_1 > 0$ and a constant C depending on f , M , J and Φ_K such that*

$$(5.6) \quad \|\rho_R\|_\infty \leq C(1 + \mu)$$

for $R > R_1$.

Proof. Let $E_R = \{\mathbf{x} \in \mathbb{R}^3 \setminus K \mid \rho_R(\mathbf{x}) > 10M\}$, $F_n = \{\mathbf{x} \in \mathbb{R}^3 \setminus K \mid 10M < \rho_R(\mathbf{x}) < n\}$ for n large. It is easy to see that the Lebesgue measure $|E_R| < \frac{1}{10}$. Choose $D \subset B_R \setminus E_R$ such that $|D| = 1$. This is possible if we choose some $R_1 > \max\{R_0, 10\}$. Now let $\gamma_1 = \frac{4}{3}$ and $\alpha_1 = \frac{5\gamma_1 - 6}{3} - \epsilon = \frac{2}{9} - \epsilon$ for some very small $\epsilon > 0$ to be determined later. Now define

$$(5.7) \quad v_1 = \begin{cases} -\rho_R^{1+\alpha_1} & \text{on } F_n \\ \int_{F_n} \rho_R^{1+\alpha_1} & \text{on } D \\ 0 & \text{otherwise} \end{cases}$$

One sees that $\rho_R + tv_1 \in W_R$ for $t > 0$ sufficiently small. Since ρ_R is a minimizer of E_μ in W_R , we have $\lim_{t \rightarrow 0^+} \frac{E_\mu(\rho_R + tv_1) - E_\mu(\rho_R)}{t} \geq 0$. Calculating the limit, we get

$$(5.8) \quad \int (A'(\rho_R) - J - B\rho_R - \mu\Phi_K)v_1 \geq 0,$$

from which it follows that

$$(5.9) \quad - \int_{F_n} v_1 A'(\rho_R) \leq \int_D v_1 A'(\rho_R) - \int_{F_n} v_1 (J + \mu\Phi_K) - \int_{F_n} v_1 B\rho_R.$$

Condition (2.10) on f implies that $A'(s) \geq C_1 \rho^{\frac{1}{3}}$ for $s > 10M$. Therefore

$$(5.10) \quad - \int_{F_n} v_1 A'(\rho_R) \geq \frac{1}{C_1} \int_{F_n} \rho_R^{\frac{4}{3} + \alpha_1}.$$

Furthermore,

$$\begin{aligned} \int_D v_1 A'(\rho_R) &\leq A'(10M) \int_{F_n} \rho_R^{1+\alpha_1}, \\ - \int_{F_n} v_1 (J + \mu\Phi_K) &\leq (\|J\|_\infty + \mu\|\Phi_K\|_\infty) \int_{F_n} \rho_R^{1+\alpha_1}, \\ - \int_{F_n} v_1 B\rho_R &\leq \|\rho_R^{1+\alpha_1}\|_{\frac{3\gamma_1}{5\gamma_1-3-3\epsilon}} \|B\rho_R\|_{(\frac{1}{\gamma_1} - \frac{2}{3} + \frac{\epsilon}{\gamma_1})^{-1}} \\ &= \|\rho_R\|_{(1+\alpha_1)\frac{3\gamma_1}{5\gamma_1-3-3\epsilon}}^{1+\alpha_1} \|B\rho_R\|_{(\frac{1}{\gamma_1} - \frac{2}{3} + \frac{\epsilon}{\gamma_1})^{-1}} \\ &= \|\rho_R\|_{\gamma_1}^{1+\alpha_1} \|B\rho_R\|_{(\frac{1}{\gamma_1} - \frac{2}{3} + \frac{\epsilon}{\gamma_1})^{-1}} \\ &\leq C\|\rho_R\|_{\gamma_1}^{2+\alpha_1}. \end{aligned}$$

Here the last step follows from lemma 3.1. Now

$$\begin{aligned}
& \int_{F_n} \rho_R^{\frac{4}{3}+\alpha_1} \\
& \leq C_1(A'(10M) + \|J\|_\infty + \mu\|\Phi_K\|_\infty) \int_{F_n} \rho_R^{1+\alpha_1} + C\|\rho_R\|_{\gamma_1}^{2+\alpha_1} \\
& \leq C_2(1+\mu)\|\rho_R\|_{1+\alpha_1}^{1+\alpha_1} + C\|\rho_R\|_{\gamma_1}^{2+\alpha_1}.
\end{aligned}$$

Since $1 + \alpha_1 < \gamma_1$, by the interpolation inequality for L^p spaces,

$$(5.11) \quad \|\rho_R\|_{1+\alpha_1} \leq C(M)\|\rho_R\|_{\gamma_1}^{\frac{4\alpha_1}{1+\alpha_1}}.$$

Hence

$$\begin{aligned}
& \int_{F_n} \rho_R^{\frac{4}{3}+\alpha_1} \\
& \leq C_3(1+\mu)\|\rho_R\|_{\gamma_1}^{4\alpha_1} + C\|\rho_R\|_{\gamma_1}^{2+\alpha_1} \\
& \leq C_3(1+\mu)\left(\int \rho_R^{\frac{4}{3}}\right)^{3\alpha_1} + C\left(\int \rho_R^{\frac{4}{3}}\right)^{\frac{3}{4}(2+\alpha_1)} \\
& \leq C_4(1+\mu)^{1+3\alpha_1} + C_4(1+\mu)^{\frac{3}{4}(2+\alpha_1)} \\
& \leq 2C_4(1+\mu)^{\frac{5}{3}}.
\end{aligned}$$

Lemma 5.1 is needed for the penultimate step, and the last step follows from the choice of α_1 . Now let n tend to infinity. Since the F_n 's increase to E_R , one gets

$$\begin{aligned}
(5.12) \quad & \int_{E_R} \rho_R^{\frac{4}{3}+\alpha_1} \leq 2C_4(1+\mu)^{\frac{5}{3}}. \\
& \int \rho_R^{\frac{4}{3}+\alpha_1} = \int_{E_R} \rho_R^{\frac{4}{3}+\alpha_1} + \int_{\rho_R \leq 10M} \rho_R^{\frac{4}{3}+\alpha_1} \\
& \leq 2C_4(1+\mu)^{\frac{5}{3}} + (10M)^{\frac{4}{3}+\alpha_1-1}M \\
& \leq C_5(1+\mu)^{\frac{5}{3}}.
\end{aligned}$$

Or,

$$(5.13) \quad \|\rho_R\|_{\frac{4}{3}+\alpha_1} \leq C_5(1+\mu)^{\frac{5}{4+3\alpha_1}}.$$

Here we assumed that we had chosen ϵ so small that

$$(5.14) \quad \frac{4}{3} + \alpha_1 = \frac{14}{9} - \epsilon > \frac{3}{2}.$$

Let $b_1(\mathbf{x}) = \frac{1}{|\mathbf{x}|}\chi_{S_1}(\mathbf{x})$ and $b_2(\mathbf{x}) = \frac{1}{|\mathbf{x}|} - b_1(\mathbf{x})$. We have $B\rho_R = \rho_R * b_1 + \rho_R * b_2$.

$$(5.15) \quad \|\rho_R * b_2\|_\infty \leq \|b_2\|_\infty \|\rho_R\|_1 \leq C.$$

Now let us pick some p between 1 and 2. Assume that we have chosen ϵ so small that the following is true

$$(5.16) \quad \frac{1}{1 - \frac{p}{5}(1+3\alpha_1)} > \frac{3}{2}.$$

Notice that since $\alpha_1 = \frac{2}{9} - \epsilon < \frac{2}{9}$, $1 - \frac{p}{5}(1 + 3\alpha_1) > 1 - \frac{p}{3} > 0$. (5.16) is equivalent to $\alpha_1 > \frac{1}{3}(\frac{5}{3p} - 1)$. Since the right hand side is less than $\frac{1}{3}(\frac{5}{3} - 1) = \frac{2}{9}$, this is possible. Now choose q satisfying $q > \frac{3}{2}$, $q < \frac{4}{3} + \alpha_1$, $q < \frac{1}{1 - \frac{p}{5}(1 + 3\alpha_1)}$. That this is possible follows from (5.14) and (5.16). Since $b_1 \in L^{q'}$ for $1 \leq q' < 3$,

$$\begin{aligned} \|\rho_R * b_1\|_\infty &\leq \|b_1\|_{q'} \|\rho_R\|_q \\ &\leq C(M) \|\rho_R\|_{\frac{4}{3} + \alpha_1}^a \end{aligned}$$

where $a = \frac{1 - \frac{1}{q}}{1 - \frac{1}{\frac{4}{3} + \alpha_1}}$, by the interpolation inequality for L^p spaces. Now it follows from (5.13) that

$$(5.17) \quad \|\rho_R * b_1\|_\infty \leq C_6(1 + \mu)^{\frac{5a}{4 + 3\alpha_1}}.$$

Combining this with (5.15), we get

$$(5.18) \quad \|B\rho_R\|_\infty \leq C_7(1 + \mu)^{\frac{5a}{4 + 3\alpha_1}}.$$

Let us calculate the exponent:

$$\begin{aligned} &\frac{5a}{4 + 3\alpha_1} \\ &= \frac{1 - \frac{1}{q}}{1 - \frac{1}{\frac{4}{3} + \alpha_1}} \frac{5}{4 + 3\alpha_1} \\ &= \frac{5(1 - \frac{1}{q})}{1 + 3\alpha_1} < p \end{aligned}$$

by the choice of q . Therefore

$$(5.19) \quad \|B\rho_R\|_\infty \leq C_7(1 + \mu)^p.$$

Now if $p \geq 3$, the same inequality is obviously true since it is already true for smaller exponents. Now let $\alpha_{l+1} = \alpha_l + \frac{1}{3}$. Define

$$(5.20) \quad v_l = \begin{cases} -\rho_R^{1+\alpha_l} & \text{on } F_n \\ \int_{F_n} \rho_R^{1+\alpha_l} & \text{on } D \\ 0 & \text{otherwise} \end{cases}$$

and repeat the previous argument, only this time using the better estimate (5.19). That gives us

$$(5.21) \quad \int \rho_R^{\frac{4}{3} + \alpha_l} \leq C_8(1 + \mu)^p \int \rho_R^{1 + \alpha_l},$$

or,

$$\begin{aligned}
\int \rho_R^{1+\alpha_{l+1}} &\leq C_8(1+\mu)^p \int \rho_R^{1+\alpha_l} \\
&\leq (C_8(1+\mu)^p)^l \int \rho_R^{1+\alpha_1} \\
&\leq (C_9(1+\mu)^p)^l \int \rho_R^{\frac{4}{3}} \\
&\leq (C_9(1+\mu)^p)^l C(1+\mu).
\end{aligned}$$

Therefore

$$\begin{aligned}
\|\rho_R\|_\infty &= \lim_{l \rightarrow \infty} \|\rho_R\|_{1+\alpha_{l+1}} \\
&\leq \lim_{l \rightarrow \infty} (C_9(1+\mu)^p)^{\frac{l}{l+1}} (C(1+\mu))^{\frac{1}{l+1}} \\
&\leq C_9(1+\mu)^p.
\end{aligned}$$

We now use this better bound on ρ_R to estimate

$$\begin{aligned}
\|\rho_R * b_1\|_\infty &\leq \|b_1\|_2 \|\rho_R\|_2 \\
&\leq C(M) \|\rho_R\|_\infty^{\frac{1}{2}} \\
&\leq C_{10}(1+\mu)^{\frac{p}{2}}.
\end{aligned}$$

Hence

$$(5.22) \quad \|B\rho_R\|_\infty \leq C_{11}(1+\mu)^{\frac{p}{2}}.$$

Since we have chosen $p < 2$, this grows at most linearly in μ . We can now repeat the previous bootstrap argument with this better estimate on $\|B\rho_R\|_\infty$. One gets

$$(5.23) \quad \int \rho_R^{\frac{4}{3}+\alpha_l} \leq C_{12}(1+\mu) \int \rho_R^{1+\alpha_l},$$

and the assertion of the lemma follows. \square

ρ_R still satisfies variational equations like (4.6) and (4.7) for $R > R_1$. From here on, we will construct a series of bounds R_n on the support of ρ_R , and a series of lower bounds μ_n for μ . Let us emphasize from the beginning that although the μ_n 's depend on f , M , Φ_K and J , the R_n 's are independent of J and μ . Also we always take $R_{n+1} \geq R_n$ and $\mu_{n+1} \geq \mu_n$.

Lemma 5.3. *There is an $R_2 > 0$ and a $\tilde{K} > 0$ such that $\lambda_R \leq 1 - \mu\tilde{K}$ for $R > R_2$.*

Proof. One first observes that if $R > R_2 > R_1$, there must be a point $\mathbf{x} \in S_{R_2}$ such that

$$(5.24) \quad \rho_R(\mathbf{x}) \leq \frac{M}{\frac{4}{3}\pi R_2^3}.$$

By (4.6),

$$(5.25) \quad \lambda_R \leq A' \left(\frac{M}{\frac{4}{3}\pi R_2^3} \right) - \mu \Phi_K(\mathbf{x}).$$

(2.10) implies that

$$(5.26) \quad \lim_{s \rightarrow 0} \frac{A'(s)}{s^{1/3}} = 0.$$

Hence

$$(5.27) \quad A' \left(\frac{M}{\frac{4}{3}\pi R_2^3} \right) = o(R_2^{-1}).$$

Pick R_2 large enough to make $A' \left(\frac{M}{\frac{4}{3}\pi R_2^3} \right) < 1$, and let $\tilde{K} = \inf_{B_{R_2}} \Phi_K > 0$. By (5.25),

$$(5.28) \quad \lambda_R \leq 1 - \mu \tilde{K}.$$

□

Lemma 5.4. *There is a $\mu_2 > 0$ such that if $\mu > \mu_2$ and $R > R_2$,*

$$(5.29) \quad B\rho_R + \mu\Phi_K \geq \frac{\mu\tilde{K}}{2} \text{ where } \rho_R > 0.$$

Proof. By (4.7) and lemma 5.3,

$$(5.30) \quad A'(\rho_R) - B\rho_R - J - \mu\Phi_K = \lambda_R \leq 1 - \mu\tilde{K}$$

where $\rho_R > 0$. Hence

$$(5.31) \quad B\rho_R + \mu\Phi_K \geq \mu\tilde{K} - 1 - J.$$

Pick $\mu_2 > \frac{2(1 + \|J\|_\infty)}{\tilde{K}}$ to get the result. □

Lemma 5.5. *There is a $\mu_3 > 0$ and an $R_3 > 0$ such that $\rho_R(\mathbf{x}) = 0$ if $R > |\mathbf{x}| > R_3$ and $\mu > \mu_3$.*

Proof. We only need to prove $B\rho_R + \mu\Phi_K < \frac{\mu\tilde{K}}{2}$ in view of (5.29). By (5.22), $\|B\rho_R\|_\infty \leq C(1 + \mu)^a$ for some $0 < a < 1$. We may choose μ_3 so large that $\frac{C(1 + \mu)^a}{\mu} < \frac{\tilde{K}}{4}$ when $\mu > \mu_3$, and R_3 so large that $\Phi_K(\mathbf{x}) < \frac{\tilde{K}}{4}$ when $|\mathbf{x}| > R_3$. The lemma then follows. □

Proof of theorem 2.3 and theorem 2.4. The argument goes exactly as before. For the constant angular velocity case just notice that the R_3 in lemma 5.5 only depends on f , M , Φ_K and not on J and μ , so we can construct a smooth increasing function

$$(5.32) \quad J(r) = \begin{cases} \frac{1}{2}\Omega^2 r^2 & \text{if } r < R_3, \\ \Omega^2 R_3^2 & \text{if } r > 2R_3, \end{cases}$$

and find a μ_0 such that a solution exists and is supported in S_{R_3} if $\mu > \mu_0$. □

6. NON-EXISTENCE FOR FAST ROTATION WITH FIXED CORE DENSITY

We now show that a solution does not exist for large enough constant rotation if the core potential Φ_K is given by the gravity of a density function ρ_K . Let us start with a few estimates.

Lemma 6.1. *Let $\rho \in L^\infty(\mathbb{R}^3)$ be a non-negative function such that $\int \rho = M$, then there is a $C > 0$ such that*

$$(6.1) \quad \|B\rho\|_\infty \leq CM^{\frac{2}{3}} \|\rho\|_\infty^{\frac{1}{3}}.$$

Proof. See [7]. □

Lemma 6.2. *Let $\rho \in L^\infty$ be a nonnegative function supported in the infinite cylinder $x_1^2 + x_2^2 \leq d^2$. Then there is a $C > 0$, such that for $x_1^2 + x_2^2 \leq d^2$,*

$$(6.2) \quad |(B\rho)_r(\mathbf{x})| \leq C\|\rho\|_\infty \left(d + \sqrt{x_1^2 + x_2^2} \right).$$

Here the subscript r denotes directional derivative in the cylindrical radial direction, even if the function under consideration is not axisymmetric.

Proof. Without loss of generality, we may assume $x_1 \geq 0, x_2 = 0, x_3 = 0$.

$$\begin{aligned} |(B\rho)_r(x_1, 0, 0)| &\leq \left| \int_{\text{Supp}\rho} \frac{\rho(x'_1, x'_2, x'_3)(x_1 - x'_1)}{\sqrt{(x'_1 - x_1)^2 + x_2'^2 + x_3'^2}^3} dx'_1 dx'_2 dx'_3 \right| \\ &\leq \int_{\text{Supp}\rho \cap \{x'_1 < x_1\}} \frac{\|\rho\|_\infty (x_1 - x'_1)}{\sqrt{(x'_1 - x_1)^2 + x_2'^2 + x_3'^2}^3} dx'_1 dx'_2 dx'_3 \\ &\leq \|\rho\|_\infty \int_{-d < x'_1 < x_1} \frac{x_1 - x'_1}{\sqrt{(x'_1 - x_1)^2 + x_2'^2 + x_3'^2}^3} dx'_1 dx'_2 dx'_3 \\ &= C\|\rho\|_\infty (d + x_1). \end{aligned}$$

The last equality follows either from a direct calculation or a simple application of the divergence theorem. □

Lemma 6.3. *Let $l = \sup \{|x_3| \mid (x_1, x_2, x_3) \in K\} + 1$, $Z = \{(x_1, x_2, x_3) \mid |x_3| \leq l\}$. Then $\Phi_K|_Z = B\rho_K|_Z \in C^{1, \frac{3}{q}}(\bar{Z})$, $\Phi_K|_{\mathbb{R}^3 \setminus Z} = B\rho_c|_{\mathbb{R}^3 \setminus Z} \in C^{1,1}(\mathbb{R}^3 \setminus \bar{Z})$.*

Proof. We first estimate $\Phi_K|_{\mathbb{R}^3 \setminus Z}$:

$$\Phi_K(x_1, x_2, x_3) = \int_{\text{Supp}\rho_K} \frac{\rho_K(x'_1, x'_2, x'_3)}{\sqrt{(x_1 - x'_1)^2 + (x_2 - x'_2)^2 + (x_3 - x'_3)^2}} dx'_1 dx'_2 dx'_3.$$

Since (x_1, x_2, x_3) is bounded away from $\text{Supp}\rho_K$, we can differentiate under the integral sign and see that

$$\begin{aligned} |D\Phi_K(x, y, z)| &\leq C \int_{\text{Supp}\rho_K} \frac{\rho_K(x'_1, x'_2, x'_3)}{\sqrt{(x_1 - x'_1)^2 + (x_2 - x'_2)^2 + (x_3 - x'_3)^2}^2} dx'_1 dx'_2 dx'_3 \\ &\leq C \int_{\text{Supp}\rho_K} \rho_K(x'_1, x'_2, x'_3) dx'_1 dx'_2 dx'_3 \\ &\leq C\|\rho_K\|_1 \\ &\leq \tilde{C}\|\rho_K\|_q^q. \end{aligned}$$

In the above inequalities, the second line is because $|x_3 - x'_3| \geq 1$, the last line is because $\text{Supp}\rho_K$ is compact. We can give a similar estimate for $D^2\Phi_K$, therefore $\Phi_K|_{\mathbb{R}^3 \setminus Z} \in C^{1,1}(\mathbb{R}^3 \setminus Z)$. As for $\Phi_K|_Z$, the Lipschitz continuity of the first derivative in a neighborhood of ∞ follows in the same way as above, whereas the Hölder continuity of the first derivative in a neighborhood of $\text{Supp}\rho_K$ follows from the standard Calderon-Zygmund inequality and the Sobolev embedding theorem. □

From now on, we assume Ω is at least 1 and use cylindrical coordinates (r, θ, z) . Let us suppose, contrary to the assertion of theorem 2.5, that there is such a ρ satisfying all the properties stated.

Lemma 6.4. $d = \sup \{r \mid (r, \theta, z) \in \text{Supp} \rho\} < \infty$.

Proof. By (2.2),

$$(6.3) \quad \frac{1}{2}r^2 \leq \frac{1}{2}\Omega^2 r^2 \leq A'(\rho) - B\rho - \Phi_K - \lambda \leq A'(\rho) - \lambda.$$

We know that $A'(s) = \int_0^s \frac{f(t)}{t^2} dt + \frac{f(s)}{s}$. It follows from (2.9) and (2.10) that $A'(\rho) \in L^\infty$ if ρ is. \square

By the expression of $A'(s)$ in the proof, we see that $A'(\rho) > 0$ iff $\rho > 0$.

Lemma 6.5. $\lambda \leq -\frac{1}{2}\Omega^2 d^2$.

Proof. Pick a sequence (r_n, θ_n, z_n) such that $\rho(r_n, \theta_n, z_n) > 0$, and $r_n \rightarrow d$. We claim that $A'(\rho)(r_n, \theta_n, z_n) \rightarrow 0$. If not, a subsequence will be bounded away from zero. Without loss of generality, we still call that subsequence $A'(\rho)(r_n, \theta_n, z_n)$. By the no trapping condition, $A'(\rho)(r, \theta_n, z_n)$ is defined for all $r > r_n$, in particular we have $A'(\rho)(d, \theta_n, z_n) = 0$. By Rolle's theorem there is an r_n^* between r_n and d such that $A'(\rho)(r_n^*, \theta_n, z_n) > 0$ and $(A'(\rho))_r(r_n^*, \theta_n, z_n) \rightarrow -\infty$. By (2.2) and the smoothing effect of B , $A'(\rho)$ is differentiable when positive. Differentiating (2.2), we get

$$(6.4) \quad (A'(\rho))_r - \Omega^2 r - (B\rho)_r - (\Phi_c)_r = 0.$$

We see a contradiction if we evaluate this expression at (r_n^*, θ_n, z_n) : the first term goes to $-\infty$ while the last three terms are bounded by lemma 6.4, lemma 6.2 and lemma 6.3 respectively. Now evaluate (2.2) at (r_n, θ_n, z_n) . By the limit of $A'(\rho)(r_n, \theta_n, z_n)$ and the positivity of $B\rho$ and Φ_K , we get the desired result. \square

Lemma 6.6. *There is a constant $C_1 > 0$, depending on Φ_K , f and M , such that $\|\rho\|_\infty \leq C_1$.*

Proof. By lemma 6.5,

$$(6.5) \quad A'(\rho) \leq B\rho + \Phi_K.$$

By (2.12), there is a $C > 0$ such that if $s > C$,

$$(6.6) \quad Cs^{\gamma-1} \leq A'(s).$$

Hence either $\rho < C$ or $C\rho^{\gamma-1} \leq B\rho + \Phi_K$. Therefore

$$(6.7) \quad C\|\rho\|_\infty^{\gamma-1} \leq C^\gamma + \|\Phi_K\|_\infty + CM^{\frac{2}{3}}\|\rho\|_\infty^{\frac{1}{3}}.$$

The last term follows from lemma 6.1. Here we have taken the liberty of using the same constant C . Now take $\epsilon > 0$ so small that $\epsilon M^{\frac{2}{3}} < \frac{1}{2}$. Since $\gamma - 1 > \frac{1}{3}$, we have

$$(6.8) \quad \|\rho\|_\infty^{\frac{1}{3}} \leq \epsilon\|\rho\|_\infty^{\gamma-1} + C(\epsilon).$$

It follows that

$$\begin{aligned} C\|\rho\|_\infty^{\gamma-1} &\leq C^\gamma + \|\Phi_K\|_\infty + \frac{1}{2}C\|\rho\|_\infty^{\gamma-1} + C(M) \\ \frac{1}{2}C\|\rho\|_\infty^{\gamma-1} &\leq C^\gamma + \|\Phi_K\|_\infty + C(M). \end{aligned}$$

The assertion now follows from the fact that $\Phi_K \in L^\infty(\mathbb{R}^3)$. \square

Lemma 6.7. *There is an $\Omega_1 > 0$ and $0 < d_0 < \frac{1}{4}$ such that if $\Omega > \Omega_1$, then $d < d_0$.*

Proof. Pick an (r, θ, z) such that $\rho(r, \theta, z) > 0$, $r > \frac{d}{2}$. Then there is an r^* between r and d such that $(A'(\rho))_r(r^*, \theta, z) \leq 0$. Evaluating (6.4) at this point, we have

$$\begin{aligned} \Omega^2 \frac{d}{2} &\leq \Omega^2 r^* \\ &\leq (B\rho)_r(r^*, \theta, z) + (\Phi_K)_r(r^*, \theta, z). \end{aligned}$$

The first term above is bounded by $2CC_1d$ by lemma 6.2 and 6.6. Noticing $(\Phi_K)_r(0, \theta, z) = 0$ by axisymmetry, the second term above is therefore bounded by $Cd^{\frac{3}{2}}$ by lemma 6.3. Now we have

$$\begin{aligned} \Omega^2 \frac{d}{2} &\leq \tilde{C}(d + d^{\frac{3}{q}}) \\ (6.9) \quad \left(\frac{\Omega^2}{2} - \tilde{C}\right)d^{1-\frac{3}{q}} &\leq \tilde{C}, \end{aligned}$$

and the assertion follows. \square

Lemma 6.8. *$\rho \in C^{0,\alpha}(\bar{S})$ for some $0 < \alpha < 1$, where S is any ball of radius $\frac{1}{2}$ whose center is on $(\mathbb{R}^3 \setminus Z) \cap x_3$ -axis, and we have $\|\rho\|_{C^{0,\alpha}(\bar{S})} \leq C_2$. Here C_2 is a constant depending on Φ_K , f and M , and Z is the region given in lemma 6.3.*

Proof. We first observe that since $A''(s) = \frac{f'(s)}{s}$, $A'(s)$ is strictly increasing. Also notice that $A'(0) = 0$. By (2.2), $A'(\rho)$ is uniformly Lipschitz continuous on $S \cap \{\rho > 0\}$ and continuous on S , hence is uniformly Lipschitz continuous on S . It is sufficient to prove

$$|\rho(\mathbf{x}) - \rho(\mathbf{y})| \leq \tilde{C}_2 |A'(\rho(\mathbf{x})) - A'(\rho(\mathbf{y}))|^\alpha,$$

or

$$(6.10) \quad A'(t) - A'(s) \geq \tilde{C}_2(t - s)^{\frac{1}{\alpha}}$$

for $0 \leq s < t \leq \|\rho\|_\infty \leq C_1$. By (2.13) there exists a $C > 0$ such that $f'(s) \geq Cs^\mu$ for $0 \leq s \leq \|\rho\|_\infty \leq C_1$. Now let $u = t - s$,

$$\begin{aligned} &(A'(t) - A'(s))(t - s)^{-\frac{1}{\alpha}} \\ &= (A'(s + u) - A'(s))u^{-\frac{1}{\alpha}} \\ &= u^{-\frac{1}{\alpha}} \int_s^{s+u} A''(\xi) d\xi \\ &= u^{-\frac{1}{\alpha}} \int_s^{s+u} \frac{f'(\xi)}{\xi} d\xi \\ &\geq u^{-\frac{1}{\alpha}} \int_s^{s+u} \frac{C\xi^\mu}{\xi} d\xi \\ (6.11) \quad &= \tilde{C}u^{-\frac{1}{\alpha}}((s + u)^\mu - s^\mu) \end{aligned}$$

If $\mu \geq 1$, (6.11) is equal to

$$\begin{aligned} & \tilde{C} \left[\left(1 + \frac{s}{u} \right)^\mu - \left(\frac{s}{u} \right)^\mu \right] u^{\mu - \frac{1}{\alpha}} \\ & \geq \tilde{C} u^{\mu - \frac{1}{\alpha}} \\ & \geq \tilde{C} C_1^{\mu - \frac{1}{\alpha}} \geq \tilde{C}_2 > 0. \end{aligned}$$

The last step is correct if we choose $\alpha < \frac{1}{\mu}$. On the other hand if $0 < \mu < 1$, (6.11) is equal to $\tilde{C} \mu \xi^{\mu-1} u^{1-\frac{1}{\alpha}}$, where ξ is between s and $s+u$. This in turn is greater than or equal to

$$\tilde{C} \mu C_1^{\mu-1} u^{1-\frac{1}{\alpha}} \geq \tilde{C} \mu C_1^{\mu-1} C_1^{1-\frac{1}{\alpha}} \geq \tilde{C}_2 > 0$$

if we choose an $\alpha < 1$. □

Lemma 6.9. *There is an $\Omega_2 > 0$ such that if $\Omega > \Omega_2$, $\rho_{\mathbb{R}^3 \setminus Z} \equiv 0$. Here Z is the region given in lemma 6.3.*

Proof. We first show that $\|B\rho\|_{C^{1,1}(\bar{S}_1)}$ is uniformly bounded, where S_1 is any ball of radius $\frac{1}{4}$ whose center is on $(\mathbb{R}^3 \setminus Z) \cap x_3$ -axis. Let S be a ball concentric with S_1 of radius $\frac{1}{2}$, then $B\rho = B(\rho\chi_S) + B(\rho\chi_{\mathbb{R}^3 \setminus S})$. The first term is bounded in $C^{2,\alpha}(\bar{S}_1)$ by lemma 6.8 and elliptic Schauder estimates. The second term is bounded in $C^2(\bar{S}_1)$ by a direct differentiation under the integral sign argument since $\mathbb{R}^3 \setminus S$ is bounded away from S_1 .

We first pick $\Omega > \Omega_1$ so that $d < d_0 < \frac{1}{4}$. Now suppose $\rho(r, \theta, z) > 0$ for some (r, θ, z) in $\mathbb{R}^3 \setminus Z$. Let us switch to Cartesian coordinates for the moment and, without loss of generality, denote this point $(x, 0, z)$ with $x \geq 0$. Let $x^* = \sup \{x \mid \rho(x, 0, z) > 0\}$. There must be a sequence $x_n \rightarrow x^*$ such that $\rho(x_n, 0, z) > 0$ and $(A'(\rho))_x(x_n, 0, z) \leq 0$, differentiating (2.2) with respect to x and evaluating at $(x_n, 0, z)$, we have

$$(6.12) \quad \Omega^2 x_n \leq -(B\rho)_x(x_n, 0, z) - (\Phi_K)_x(x_n, 0, z).$$

Taking limit as $n \rightarrow \infty$, we get

$$(6.13) \quad \Omega^2 x^* \leq -(B\rho)_x(x^*, 0, z) - (\Phi_K)_x(x^*, 0, z).$$

If there was an $x_0 \in [0, x^*)$ such that $\rho(x_0, 0, z) > 0$ and $(A'(\rho))_x(x_0, 0, z) \geq 0$, we would have

$$(6.14) \quad \Omega^2 x_0 \geq -(B\rho)_x(x_0, 0, z) - (\Phi_K)_x(x_0, 0, z).$$

Subtracting (6.14) from (6.13), we get

$$(6.15) \quad \begin{aligned} \Omega^2(x^* - x_0) & \leq ((B\rho)_x(x_0, 0, z) - (B\rho)_x(x^*, 0, z)) \\ & \quad + ((\Phi_K)_x(x_0, 0, z) - (\Phi_K)_x(x^*, 0, z)). \end{aligned}$$

The first term on the right hand side is bounded by $C(x^* - x_0)$ because $B\rho$ is uniformly bounded in $C^{1,1}(\bar{S}_1)$ as indicated above, while the second term is bounded by $C(x^* - x_0)$ because of lemma 6.3. Hence (6.15) becomes

$$(6.16) \quad \Omega^2(x^* - x_0) \leq 2C(x^* - x_0),$$

which is impossible if we choose $\Omega_2 > \max\{\Omega_1, 2C\}$. Therefore such an x_0 does not exist. This in particular implies that there is no $x \in [0, x^*)$ for which $\rho(x, 0, z) = 0$, which then implies that $\rho(0, 0, z) > 0$ and $(A'(\rho))_x(0, 0, z) < 0$. But exactly the same argument in the $-x$ direction would imply $(A'(\rho))_x(0, 0, z) > 0$. This contradiction indicates that there is no such (r, θ, z) in the first place, and the assertion is therefore true. \square

We are now ready to give

Proof of theorem 2.5. By lemma 6.9, $\text{Supp}\rho$ is uniformly bounded in the z direction. Recall from lemma 6.3 that this bound is given by l ,

$$\begin{aligned} M &= \int_{\text{Supp}\rho} \rho \\ &= \int_{|z| \leq l, r \leq d} \rho \\ &\leq \|\rho\|_{\infty} 2\pi d^2 l \\ &\leq 2C_1 \pi d^2 l. \end{aligned} \tag{6.17}$$

Therefore

$$d \geq \sqrt{\frac{M}{2C_1 \pi l}}. \tag{6.18}$$

Compare this with (6.9), we get

$$\left(\frac{\Omega^2}{2} - \tilde{C}\right) \sqrt{\frac{M}{2C_1 \pi l}}^{1-\frac{3}{q}} \leq \tilde{C}, \tag{6.19}$$

which is clearly false if we choose $\Omega_0 > \Omega_2$ sufficiently large. This contradiction indicates that such a solution ρ does not exist. \square

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